Integrable Structure of Interface Dynamics

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We establish the equivalence of 2D contour dynamics to the dispersionless limit of the integrable Toda hierarchy constrained by a string equation. Remarkably, the same hierarchy underlies 2D quantum gravity.

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1. Laplacian growth. Contour dynamics takes place in many physical processes, where an interface moves between two immiscible phases. The key example of interface dynamics to illustrate the main result of this work is the Laplacian growth (LG) [1]. This process is dissipative, unstable, ubiquitous (applications range from oil/gas recovery to tumor growth), and universal: a steady self-similar pattern appears governed by scaling laws, most of which still yet to be derived [1,2].

In this paper we show that an arbitrary interface dynamics has an integrable structure which is the same as the one that underlies models of 2D quantum gravity. This structure links the interface dynamics, and especially LG, with other branches of theoretical physics, where scaling laws are also expected [17].

2. To be specific, we will speak about Hele-Shaw flow [1]: a viscous fluid (oil) and a non-viscous fluid (water) are confined in a narrow gap between two parallel plates. The interior water domain, D_+, is surrounded by an exterior oil domain, D_-, occupying the rest of the plane. Water is supplied from the origin and pushes the oil/water interface, $\mathcal{C}(t)$. Both liquids are incompressible, so oil is extracted at infinity at the same rate q as water is supplied.

– the normal velocity of the interface is $V_n = -\partial_n p$ (the D'Arcy law); the pressure p is kept constant ($p = 0$) inside the water domain $D_+(t)$ and on the interface (surface tension and viscosity of the water are neglected); and pressure is a harmonic function, $\nabla^2 p = 0$, inside the oil domain $D_-(t)$, while $p \to -(q/2\pi)\log\sqrt{x^2 + y^2}$ at infinity.

This (idealized) LG problem has an important prop-erty: The harmonic moments of the oil domain $C_k = \int_{D_-(t)} z^{-k} dx\,dy$ ($k = 1, 2, \ldots$, and $z = x + iy$)[1] do not change in time, while the area of water domain, grows linearly in time [3]. The proof:

$$\frac{dC_k}{dt} = \oint_{\mathcal{C}(t)} \frac{V_n\,d\mathcal{C}}{z^k} = \oint_{\mathcal{C}(t)} (p\,\partial_n z^{-k} - z^{-k}\partial_n p)\,d\mathcal{C},$$

because $V_n = -\partial_n p$ and $p = 0$ along the $\mathcal{C}(t)$. By virtue of the Gauss' theorem, it equals

$$= \int_{D_-(t)} \nabla\,(p\,\nabla z^{-k} - z^{-k}\nabla p) dx\,dy = -q\delta_{k,0}.$$

This property may be used as the definition of the idealized LG problem:

– To find the form of the domain whose area increases while all harmonic moments remain fixed.

This problem is known to be ill-defined [1]. For almost all sets of harmonic moments, the boundary develops cusp-like singularities in finite time (area) [1]. Once a singularity occurs, the idealized LG model is no longer valid. Surface tension, omitted above, stabilizes the growth and simultaneously ruins the conservation of harmonic moments. Simulations and experiments show that different mechanisms of regularization of singularities (surface tension, lattice etc.) exhibit the same self-similar pattern [1,2]. This suggests a fixed point (or points) in the space of harmonic moments, which correspond to observed stable patterns. To identify the fixed points and their scaling properties is the challenge of the growth phenomena.

Approach to a fixed point requires a change of *all* moments. This is the question we address in this paper.

We present the set of differential equations which describe the evolution of the domain under a variation of *all* harmonic moments. This prompts to a connection with the inverse potential problem [4]: to restore the shape of

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[1]C_1 and C_2 are finite: the divergence as $|z| \to \infty$ cancels by integration over $\arg z$.

a body from a given Newtonian potential of a uniform mass distribution inside the body.

It remains to be seen whether these equations help to describe the pattern of growth; however, they reveal the integrable structure of the growth problems. We will show that the equations describing the evolution of a domain form an integrable hierarchy. Moreover, the very same hierarchy emerges in $c = 1$ string theory and topological gravity [5], and in 2-matrix models [6]. It is the dispersionless limit of the 2D Toda hierarchy [7] constrained by the so-called string equation [8].

3. To proceed further we need some known facts about the Schwarz function. (See e.g., [9]). An equation for a curve, $F_\mathcal{C}(x,y) = F_\mathcal{C}(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) = 0$, can be resolved (at least locally) with respect to one of the complex variables, say $\bar{z} = x - iy$. The result, $\bar{z} = S(z)$, is called the *Schwarz function* of the curve $\mathcal{C}$ (see e.g., [9]):
(a) $S(z)$ is a unitary operation: $\bar{S}(S(z)) = z$;
(b) The unit vector tangential to the curve is $dz/dl = dz/\sqrt{dzd\bar{z}} = (d\bar{z}(z)/dz)^{-1/2} = 1/\sqrt{S_z} = \sqrt{S_{\bar{z}}}$;
(c) For simple analytic curves, the Schwarz function can be analytically continued to some strip-like domain containing the curve.

The function $S(z)$ can be decomposed into a sum of two functions $S^{(\pm)}(z)$ that are regular in $D_\pm$: $S(z) = S^{(+)}(z) + S^{(-)}(z)$. Under the condition $S^{(-)}(\infty) = 0$ this decomposition is unique. The functions $S^{(\pm)}(z)$ can be represented by a Taylor series convergent near the origin (which is assumed to be in D_+) and near infinity in D_-: $S^{(+)}(z) = \sum_{k=0}^{\infty} S_k z^k$, $S^{(-)}(z) = \sum_{k=1}^{\infty} S_{-k} z^{-k}$. The coefficients $S_{\pm k}$ are nothing but harmonic moments of the exterior, D_-, and the interior, D_+, domains:

$$C_{\pm k} = \mp \int_{D_\mp} z^{\mp k} dx dy = \oint_{\mathcal{C}(t)} \frac{S(z)\, dz}{2i z^{\pm k}} = \pi S_{\pm k - 1} . \quad (1)$$

In other words, $S^{(\pm)}(z)$ is the gradient of the Newtonian potential created by matter uniformly distributed in the interior (exterior) of $\mathcal{C}$. In these terms the idealized LG problem implies that $S^{(+)}$ does not vary in time and $\pi S_{-1} = $ (Area of D_+) grows linearly in time.

The Schwarz function is closely related to conformal maps. Let $\phi(x,y)$ be the function harmonically conjugate to $2\pi p(x,y)/q$. Then $w = e^{-2\pi p/q + i\phi}$ univalently maps the oil domain to the exterior of the unit circle. This map sends $w = \infty$ to $z = \infty$. Let us write

$$z(t,w) = r(t)\, w + \sum_{k=0}^{\infty} u_k(t)\, w^{-k} \quad (2)$$

for the inverse map, where r is chosen to be real, so the map $z(w)$ is unique. This map and the map to the complex conjugate domain $\bar{D}_-$,

$$\bar{z}(t, w^{-1}) = r(t)\, w^{-1} + \sum_{k=0}^{\infty} \bar{u}_k(t)\, w^k , \quad (3)$$

resolve the unitary condition for the Schwarz function and give it the following interpretation. If w is the image of a point z, then $S(z)$ is the complex conjugate pre-image of w^{-1}: $S(z) = \bar{z}(w^{-1}(z))$.

4. The idealized LG problem has an instructive form in terms of the Schwarz function:

$$\partial_t S = \frac{q}{\pi} \partial_z \log w. \quad (4)$$

To derive (4) (following [10]), we differentiate $\bar{z}(t, w^{-1}) = S(t, z(t,w))$. We get $\partial_t \bar{z}(t, w^{-1}) = \partial_t S(t,z) + \partial_z S(t,z)\partial_t z(t,w)$ and, by virtue of (b), $V_n = \text{Im}(\bar{z}_t z_l) = S_t/(2i\sqrt{S_z})$. From $\log w(z) = -2\pi p(x,y)/q + i\phi(x,y)$ and $p = const$ along $\mathcal{C}(t)$, we conclude that $-\partial_n p = q w_z/(2\pi i w \sqrt{S_z})$. Since $V_n = -\partial_n p$, we obtain (4). From now on we set $q = \pi$ by making a proper time rescaling.

The equation (4) written in terms of the conformal maps (2), (3) has the form [1]:

$$\{z(t,w),\, \bar{z}(t, w^{-1})\} = 1, \quad (5)$$

where we define the Poisson bracket by $\{f, g\} \equiv w(\partial_w f \partial_t g - \partial_t f \partial_w g)$. On comparing powers of w for both sides of (5), we get a set of equations for the coefficients $u_k, \bar{u}_k$ of the Laurent series, (2) and (3), with fixed C_k. Eq. (5) suggests the Hamiltonian structure of the problem: $z(t,w)$, $\bar{z}(t, w^{-1})$ and $\log w$, t are canonical pairs.

5. Consider the function $\Omega(z)$ defined on the curve by

$$\Omega(z) = \frac{|z|^2}{2} + 2i A(z), \quad z \in \mathcal{C} , \quad (6)$$

where we have separated the real and imaginary parts. Here $A(z)$ is the area of the sector enclosed by $\mathcal{C}$ and bounded by the ray $\arg z$ and some fixed reference ray. Just like the Schwarz function, this function can be analytically continued within a strip containing the curve. Indeed, writing (6) in terms of $S(z)$, we have

$$\Omega(z) = zS(z)/2 + 2i\int^z [S(z')dz' - z'dS(z')]/4i = \int^z S(z')dz'$$

$$= \sum_{k=1}^{\infty} C_k z^k/\pi + t \log z - v_0/2 - \sum_{k=1}^{\infty} C_{-k} z^{-k}/\pi, \quad (7)$$

where v_0 does not depend on z. As is seen from the above definition, Ω is defined up to a purely imaginary z-independent term. We fix it by requiring v_0 to be real.

The function Ω is the generating function of the canonical transformation $(\log w, t) \to (z, \bar{z})$. Indeed, from (7) we have $S(z) = \partial_z \Omega(z)$ and, by virtue of (4), $\log w = \partial_t \Omega$. Therefore, the differential $d\Omega = S(z)\, dz + \log w\, dt$ encodes the LG equations (4) or (5).

6. Now we extend the differential $d\Omega$ to include variations of all higher moments. For $k \geq 1$, let us denote

$$t_k = \frac{C_k}{\pi k}, \ H_k = \frac{\partial \Omega(z)}{\partial t_k}, \ \bar{H}_k = -\frac{\partial \Omega(z)}{\partial \bar{t}_k} . \quad (8)$$

Then the multi-time Hamiltonian system is defined as

$$d\Omega = S(z)\,dz + \log w\,dt + \sum_{k=1}^{\infty}(H_k dt_k - \bar{H}_k d\bar{t}_k)\,.$$

Thus, the flows with respect to the "times" t_k are

$$\partial_k z = \{z, H_k\}\,, \quad \partial_{\bar{k}} z = \{z, \bar{H}_k\}\,, \qquad (9)$$

where $\partial_k = \partial/\partial t_k$, $\partial_{\bar{k}} = \partial/\partial\bar{t}_k$, and $\{,\}$ is the canonical Poisson bracket introduced above. These equations are consistent due the symmetry relations $\partial_l H_k(z) = \partial_k H_l(z)$ which follow from (8). In terms of w, these conditions have the form of the zero-curvature equations:

$$\partial_k H_l(w) - \partial_l H_k(w) = \{H_k(w), H_l(w)\}\,. \qquad (10)$$

We now proceed to calculate the Hamiltonians. Below we will prove that in addition to (8), for z on the curve, the Hamiltonians can be equivalently defined as

$$H_k = -\,\partial_k\bar{\Omega}(\bar{z})\,. \qquad (11)$$

(The derivative is taken at fixed $\bar{z}$.) Then (7) gives us

$$H_k = z^k - \partial_k v_0/2 - \sum_{l=1}^{\infty}\partial_k v_l\,z^{-l}/l = \qquad (12)$$

$$= \partial_k v_0/2 + \sum_{l=1}^{\infty}\partial_k \bar{v}_l\,\bar{z}^{-l}/l, \qquad (13)$$

where we set $v_k = C_{-k}/\pi$. Eq.(12) implies that the Laurent expansion of the H_k at $w = 0$ does not contain powers of w higher than w^k. Moreover, all non-negative powers of w come from the first two terms of Eq.(12). In turn, Eqs. (13) and (3) imply that H_k does not contain negative powers of w. Altogether they mean that H_k is a polynomial in w of degree k. It reads

$$H_k(w) = (z^k(w))_+ + \frac{1}{2}(z^k(w))_0\,. \qquad (14)$$

The symbol $(f(w))_+$ means a truncated Laurent series, where only terms with positive powers of w are kept; $(f(w))_0$ is the constant (w^0) part of the series.

It remains to prove Eq. (11). We first notice that $\partial_j \mathrm{Re}\,\Omega(z) = 0$ if z belongs to the curve. This property is proved by differentiating the real part of (6), $\Omega(z) + \bar{\Omega}(\bar{z}) = |z|^2$. The analytic continuation away from the curve gives $\Omega(z) + \bar{\Omega}(S(z)) = zS(z)$. Taking the partial derivative with respect to t_j and restricting the result to the curve again, we get:

$$\partial_k\Omega(z) + \partial_k\bar{\Omega}(\bar{z}) + \partial_k S(z)\,\partial_{\bar{z}}\bar{\Omega}(\bar{z}) = z\partial_k S(z)\,. \qquad (15)$$

But the r.h.s. and the last term in the l.h.s. of (15) are equal since $z = \bar{S}(\bar{z}) = \partial_{\bar{z}}\bar{\Omega}(\bar{z})$. Thus (15) reads

$$\partial_k[\Omega(z) + \bar{\Omega}(\bar{z})] = 0\,, \qquad (16)$$

where z belongs to the curve. Eq. (11) follows by virtue of (8). Now we see that H_k and $\bar{H}_k$ (defined as in (8)) are indeed complex conjugates on the curve.

Eqs. (9) or alternatively (10) together with (14), (2), and (5) provide an algorithm generating equations for the coefficients of the conformal map (2). Two first Hamiltonians are $H_1 = rw + u_0/2$, and $H_2 = r^2 w^2 + 2ru_0 w + ru_1 + u_0^2/2$. The first equation of the hierarchy is

$$\partial_{1\bar{1}}^2 \varphi = \partial_t \exp(\partial_t\varphi), \qquad (17)$$

where $r^2 = \exp(\partial_t\varphi)$. One can see from $\partial_t\Omega = \log w$ that φ is the constant term in the expansion (7): $\varphi = v_0$.

7. The unitarity condition (a) for the Schwarz function and the properties (6), (16) of the generating function Ω (which actually follow from the unitarity) impose important relations among the harmonic moments of any smooth simply connected domain and the harmonic moments of its complement. First, from (6) one can derive the following sum rules:

$$\sum_{k\geq 1} kt_k v_k = \sum_{k\geq 1} k\bar{t}_k\bar{v}_k\,, \quad \bar{v}_1 = tt_1 + \sum_{k\geq 2} kt_k v_{k-1}\,. \qquad (18)$$

Second, there are symmetry relations for derivatives of the harmonic moments $v_k = C_{-k}/\pi$ of the interior domain D_+ with respect to the (rescaled) harmonic moments t_j of the exterior domain: $\partial_j v_k = \partial_k v_j$, $\partial_{\bar{j}} v_k = \partial_k\bar{v}_j$, $\partial_t v_k = \partial_k v_0$. The proof: it follows from (13) that $\oint_{\mathcal{C}} H_j dH_k = 0$ for all j, k, then it is easy to see that

$$\partial_j v_k = \oint_{\mathcal{C}} z^k dH_j/2\pi i = \oint_{\mathcal{C}} z^j dH_k/2\pi i = \partial_k v_j\,.$$

This implies that for each analytic curve $\mathcal{C}(t, t_j)$ there exists a real function (prepotential) $F(t, t_j, \bar{t}_j)$ such that

$$v_j = \partial_j F\,, \quad \bar{v}_j = \partial_{\bar{j}} F\,, \quad v_0 = \partial_t F\,. \qquad (19)$$

This function determines $H_k(z)$ via (12).

8. Equations (9), (14) are familiar in the soliton literature as the dispersionless limit [7] of the 2D Toda hierarchy. Dispersionless hierarchies of this kind [11] are extensions of the integrable equations of hydrodynamic type [12] to the multidimensional case. Many special solutions were found in [13]. Eq. (5) is known as the string equation [8]. (See also [14] for more recent developments in the 2D Toda hierarchy.) To make the contact we now review, following [7], the standard setup of the 2D Toda hierarchy and its dispersionless limit.

The 2D Toda hierarchy is usually introduced by means of two difference Lax operators:

$$L = r(t)e^{\hbar\partial_t} + \sum_{k=1}^{\infty} u_k(t)e^{-k\hbar\partial_t},$$

$$\bar{L} = r(t-\hbar)e^{-\hbar\partial_t} + \sum_{k=1}^{\infty} \bar{u}_k(t)e^{-k\hbar\partial_t}, \qquad (20)$$

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where r, u_k and $\bar{u}_k$ are functions of t and of two sets of independent parameters t_k, $\bar{t}_k$, $k > 0$. These functions obey the Lax-Sato equations:

$$\hbar\partial_{t_k} L = [L, H_k], \quad \text{where } H_k = (L^k)_+ + (L^k)_0/2,$$
$$\hbar\partial_{\bar{t}_k} \bar{L} = [\bar{H}_k, L], \quad \text{where } \bar{H}_k = (\bar{L}^k)_- + (\bar{L}^k)_0/2. \quad (21)$$

The symbol $(L^k)_\pm$ means the part of the operator that consists of positive (negative) powers of the shift operator $e^{\hbar\partial_t}$, and $(L^k)_0$ is the part that does not contain the shift operator. The first equation of the hierarchy is the familiar 2D Toda equation:

$$\partial_{1\bar{1}}^2 \varphi(t) = e^{\varphi(t+\hbar)-\varphi(t)} - e^{\varphi(t)-\varphi(t-\hbar)}, \quad (22)$$

where $r^2(t) = e^{\varphi(t+\hbar)-\varphi(t)}$. It is also customary to consider the Orlov-Shulman operators [15]

$$M = \sum_{k=1}^\infty k t_k L^k + t + \sum_{k=2}^\infty v_k L^{-k},$$
$$\bar{M} = \sum_{k=1}^\infty k \bar{t}_k \bar{L}^k + t + \sum_{k=2}^\infty \bar{v}_k \bar{L}^{-k}, \quad (23)$$

where $v_k = v_k(t, t_j, \bar{t}_j)$, $\bar{v}_k = \bar{v}_k(t, t_j, \bar{t}_j)$ are functions such that the operators obey the conditions $[L, M] = \hbar L$, $[\bar{L}, \bar{M}] = -\hbar\bar{L}$. These operators satisfy the following linear equations: $L\Psi = z\Psi$, $\partial_k\Psi = H_k\Psi$, $\hbar z\partial_z\Psi = M\Psi$, and similarly for the bar-operators acting on $\bar{\Psi}$.

9. One particular solution of the 2D Toda hierarchy describes 2-matrix models. Consider the integral over two hermitian $N \times N$ matrices

$$\tau = \int e^{Ntr(M\bar{M}+\sum_{k>0}(t_k M^k + \bar{t}_k \bar{M}^k))} dM d\bar{M} \quad (24)$$

(the partition function). It has been shown that this integral is the τ-function for a special solution of the 2D Toda hierarchy with $\hbar = 1/N$ [16]. The solution is selected by the string equation $[L, \bar{L}] = \hbar$. The coefficients v_k of the M-operator are given by $v_k = \partial_k \log \tau = <M^k>$, where $< \cdots >$ means an average over matrices with the weight (24). A scaling behavior of a proper large N limit of the matrix model is expected to describe 2D gravity [17].

The dispersionless hierarchy is obtained in the limit $\hbar \to 0$. In this limit, the shift operator $e^{\hbar\partial_t}$ is replaced by a classical variable w, the Lax operators are substituted by their eigenvalues $L \to z(w)$ and $\bar{L} \to \bar{z}(w^{-1})$, and the operator, $L^{-1}M$, becomes a function $S(z)$. At the same time all commutators are replaced by the Poisson brackets with the symplectic structure $\{w, t\} = w$, so $[L, \bar{L}] = \hbar$ turns into (5). Eq. (17) is the $\hbar \to 0$ limit of Toda's equation (22). The wave function Ψ is replaced by $e^{\Omega/\hbar}$, where Ω is the generating function of the canonical transformation $(\log w, t) \to (z, \bar{z})$. At last, the τ-function is $\tau = e^{F/\hbar^2}$ as $\hbar \to 0$, where the function F is the prepotential introduced in (19).

10. To summarize, comparing the semiclassical limit of the Toda Eqs.(20), (21), and the string equation, with Eqs.(2), (3), (9), (14), (5) of an arbitrary interface dynamics, we find an exact equivalence between them. This is the main result of this work. The sum rules (18) for the harmonic moments are nothing else but (a part of) the W-constraints for the τ-function. It also follows from above that the interface dynamics is equivalent to the $N \to \infty$ planar limit of the matrix model (24): the (logarithm of) the partition function of the latter is the prepotential function F (19).

It is tempting to understand what do the robust scaling behavior observed in variety of growth problems [2] and a scaling behavior of 2D gravity [17], [18], have in common.

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